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Rectifying misconceptions of linear algebra made by students of tertiary education regarding the concept of equivalence: affordances offered by a multi-representational computational environment

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Abstract
In the presented herein work, the results of a continuing research project are discussed, regarding the current academic period and involving students of the University of West Attica, Greece. The aim of this research is to optimize the teaching of linear algebra, especially of the linear equations systems, by employing a set of instructional interventions suitably designed for this purpose. The designing of these interventions is conducted according to the logic of both the Three Worlds of Mathematics by D. Tall and the Experiential Learning Model (ELM) by D. Kolb. The computational environment of Mathematica, especially its computable document formats (CDFs), is used, in order to offer a multi-representational dimension of the mathematical concepts in question.

Keywords: CDFs, multi-representational environments, systems of linear equations, equivalence, three worlds of mathematics, experiential learning model

1. Introduction

Linear algebra is one of the first courses of advanced mathematics, taken by freshmen in tertiary education, which, together with analysis, leads them into a higher level of abstract thought than the one they encountered in school mathematics. The teaching and learning of linear algebra is quite a demanding task from a cognitive point of view [1], and it still remains a difficult subject to comprehend [2, 3]. During the first stages of their studies, the students face certain formal mental abstractions, in a way they have not met previously. The aim is to induce the transformation of their cognitive patterns and to turn them into active and creative solvers of problems, by developing their individual abilities and skills. The difficulty in achieving this aim is that the students with the traditional approach apparently learn the matrix algebra, take and pass their examinations, without knowing in depth the most fundamental concepts of the subject, while very few of them conquer more advanced knowledge, which requires an understanding at higher levels of abstract thinking [4, 5, 6, 7]. In most cases, the mistakes of the students are treated in a totally negative way, without using them in a positive, constructive way as an opportunity for reflection, investigative thinking and active construction of the pursued conceptual change [8]; often, the instructors ignore the part of their responsibility for the effectiveness of the learning process. The need to invent novel models of teaching – a basic element of which should be, inter alia, the understanding of the modes of thinking and of the particular learning difficulties of the students – in order to design pedagogically suitable learning scenarios and to create advanced, from a pedagogical point of view, learning environments, is the aim of the described research, which is still under way.

2. Theoretical framework

In the present research, the following theoretical tools have been used:

1. *The qualitative organization of mathematical thinking, known as "Three Worlds of Mathematics"* by David O. Tall [9, 10, 11]. These Three Worlds mention:

1. *The embodied world:* the quality of knowledge, acquired through the learning experience as an action of the person upon the objects, and the perception he/she has of these objects, to which contribute the informal types of learning, as well as the interventions of the representations of the senses that assist the formation of personal mental images of the individual about an object. This world is extended to the respective mental constructions that result from the abstraction, but it always remains connected to the action and the output of the senses, from which it originates. These mental images may not be compatible with the "official" version of the mathematical knowledge. The learning individual benefits, if he/she participates in the building of the learning procedure, during which the world of this individual is transformed via conceptual change.
2. *The operational-symbolic mathematical world*: The individual in this world can manipulate mathematical objects and symbols at the symbolic-*proceptual* level and, in case he/she advances further, he/she utilizes the role of symbols in arithmetic, algebra and symbolic analysis, by combining their double nature: as a process and as a concept *(process-concept: procept)*. This is the world of mathematical processes and also symbol processing and handling.
3. *The axiomatic (formal) world:* This is the favorite world of the person who approaches mathematics in the formal manner, starting from selected axioms and fundamental concepts, and, through a series of logical

conclusions, being led to the proof of the rest of mathematics. In this world, the propositions are true, because they can be proved, based on axioms and theorems. Due to the rigor and the formalism it requires, this world presents the greatest difficulty with respect to the learning procedure of the students.

The role of language is very important in all three worlds, since through verbal communication and negotiation, the concepts being taught are placed into their proper frame. In any case, however, the conceptual knowledge is built in a better and more complete way, when the student operates in all three worlds and smoothly moves from one to the other and when all three worlds get interconnected, in order to construct a concept or notion as a whole.

2. ***The Experiential Learning Model of Kolb*****:** David A. Kolb [12] proposed a model of empirical learning in four stages, which can be applied for all persons, being educated, since it gives them the opportunity to engage in activities that match the style of learning they prefer. More importantly, this catering of variable needs – when repeated in the manner of a Bruner spiral (Fig. 1) – repeatedly affords the individual of familiar occasions for boarding the learning process, wherever it suits them [13, 14, 15]. The stages are:

Stage 1: Experience of the concrete, Feeling-living experience, Motivation;
Stage 2: Observation of and reflection on that experience, Discussion;
Stage 3: Abstract thinking, Conceptualization, Generalization, Theorization,
and finally
Stage 4: Active experimenting, Action, Production of new learning experiences.

In order to track the learning process, the way of thinking and acting of the person that delves into mathematical objects and concepts, the APOS model [16] is used as a diagnostic tool. The four modes of mathematical knowledge of this model are:

a. action-oriented or procedural knowledge,
b. process-oriented knowledge,
c. object-oriented knowledge, and
d. conceptual or schema-knowledge.

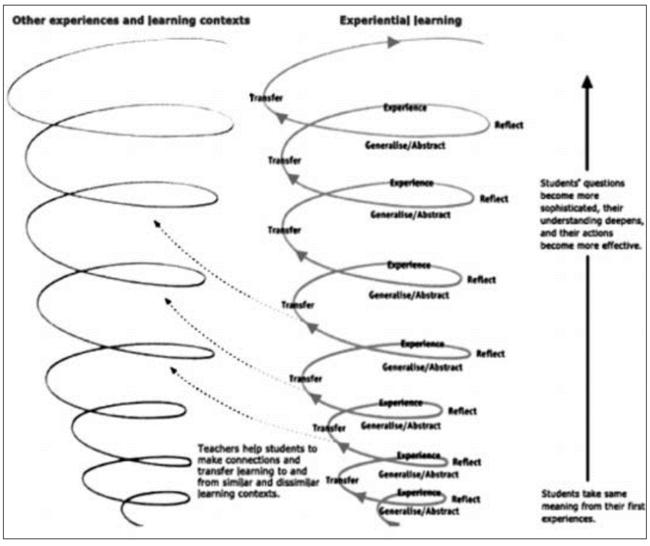

Fig 1: The Spiral of Experiential Learning. Source: [17].

2. Research questions

1. What kind of learning obstacles and difficulties are experienced by the students, in understanding concepts in the field of linear algebra, and especially that of equivalence?
2. How the quality of the students' mathematical thinking evolves on the basis of the "Three Worlds" theory (of Tall) and the Experiential Learning Model (of Kolb)?

3. Methodology

Because of the qualitative nature of the research questions, the case study within the framework of action research was adopted as a methodological approach. The data collection in a case study seldom takes place as a separate phase; instead, it takes place throughout the period of the study. This applies even more in a case study that is at the same time a teaching intervention: in the best tradition of action research, the collection of data has to follow the development of the intervention. As the intervention was structured in the form of successive overlapping Kolb cycles in each session, introducing new experiences about newly taught related concepts, there were reflection activities in progress upon previous experience, efforts to draw conclusions and theories upon them, or experimentations – helping to turn the cycles into a spiral – during which the reactions of students and the experiment's influence on the teaching process were worthy of recording.

In the present teaching intervention:

1. The researchers had at the same time the role of the instructors, while the intervention had to do with all the students who were involved into learning cycles, both individually and as a team, and participated in all the phases of the intervention equally and without differentiations. Both the intervention and the study functioned collaboratively.
2. There was a coupling of search and action with the theory and practice, while there also was practical use of the results, by implementing strategies of action that afterwards were critically evaluated and modified. In this way, a dialectic investigation of theory and practice was attempted, an approach that is a suitable way to develop both of them, within a frame of continuous practical and theoretical transformations.
3. There was a basic scheduling on a set of phases that pertained to the research questions, in such a way that they would not define the path of the study, but simply orient it, functioning as a starting point for planning. It was an open cyclic process, during which the participants were acting, having understanding modification and improvement as a goal; it was a spiral that crossed Tall's Three Worlds of Mathematics.

The described intervention was initially designed as complementary to the course, and it was tested with volunteers from among the students in the laboratory, not in the class hours, with considerable success. A realistic intervention, however, cannot be outside of the scheduled hours of the main course. Therefore, this year it was modified in order to be used during theoretical instruction, which introduced and clarified the related concepts of equivalent systems, the unique solution and its geometrical representation, but also more generally the geometrical meaning and solution of a system of equations.

The CDFs (Computable Document Formats) of

Mathematica software, as a visualizing technique, are used in order to mediate for the evolution of the students' thinking in the direction of increased abstraction and improvement of their intuitive/geometrical perception. Mathematica is one of the preferred software tools for supporting linear algebra teaching, for some time now [18] and CDFs offer further usability. Having previously demonstrated [15, 19] that it is possible to achieve instrumental genesis and even instrumentalization with such tools, it was only sought here to demonstrate the functionality of specific choices at the level of comprehension of a concept, such as the concept of equivalence.

4. Didactic Intervention

***1st Activity*:** When prompted to "give an example of a linear system of equations 2 by 2 that has no solution" in any way they liked, the students wrote down correctly an inconsistent system, two lines parallel to each other in the form of equations; thus they demonstrated that they comprehend the rules with the required ratios between the coefficients of the variables, showing preference to the algebraic way of thinking, since none of the students used the geometrical representation.

Investigating further the cognitive schemes of the students, the following action was set: "Starting from the inconsistent linear system you have devised, give its equivalent expression in the form of a matrix equation" (the form AX=B) and "solve the linear system in any way you like". The students, although in some cases added a formal observation ("the matrix A should be invertible", but also "therefore its determinant is nonzero"), they were not able to directly transform their example of inconsistent linear system in its matrix form. While they knew the matrix multiplication rules, they did not have the necessary familiarity, in order to be able to produce at a cognitive level of process and object (*precept*) what they were asked for; instead, they remain at the level of the automatic actions. Directly expressing it in a system in its matrix form would require a mental leap, so that the students, through reflection on their actions, could connect the structural elements of the mathematical object on a meaningful novel comprehensive structure.

In the inverse problem, that is when they were given a matrix equation AX = B, they managed to write down the algebraic equations of the resulting system, but only after they did the multiplications of the elements in the left-hand side of the matrix equation and they (according to the equality rule for two matrices) equated the respective elements of the two matrices. In other words, there was no immediate writing down of the respective linear system of equations, a fact indicating that the cognitive scheme of the matrix multiplication process had not been encapsulated into a scheme of object knowledge in the context of the APOS framework.

The students did not conceive the form of matrix equation as an alternative form of symbolic representation of a system of linear equations. That is, as far as the linear systems are concerned, their symbolic world according to Tall (with the system and matrix expressions) was fragmented, only partly recognizable and not unified. As for the equivalence of the two expressions or symbolic forms of the linear systems (the Gaussian elimination being applied to the matrix form) it has been noticed that they did not use in their answers the term "equivalence/equivalent". To the questions:

- "What do you have to say about the expressions we investigate?" and
- "Which are the relations between the two expressions?",

the students answered with expressions such as:

- "what do you mean which are the relations?" or
- "I can find the solutions to the system either with the one method or with the other",

until finally someone said "therefore, they are equivalent expressions"; only then the other students agreed with this statement.

More specifically, the students knew about the Gaussian elimination for a 2×2 system, and no problems emerged relative to its application, however, as they solved the system, using the augmented matrix, pre-existing cognitive models about the behavior of this procedure in the case of inconsistent systems resulted in misunderstandings that finally led to erroneous conclusions. One of these cases was the notion of equivalent systems. The existence of such models was apparent, since students typically said:

- "I did not relate the Gaussian criterion to the equivalence of the systems",
- "the Gaussian criterion is that when you have a row with zeros and in its last column you have 1, then the system is inconsistent",
- "the process in this way … appears usually in the last row",
- "we try to make it to appear in the last row",
- "we follow a method and … it makes it appearing in the last row",
- "since we want to end up with a diagonal matrix …".

It becomes evident that the students, equipped with their previously acquired knowledge, did not attain the dexterity to treat the linear systems in a dynamic manner, as a world transforming through processes into equivalent expressions that do not lose their initial characteristics or properties. They faced the final expressions as new objects, not realizing throughout the process that equivalence to the original characterizes them, and this incomplete model trapped them into practices that create problems in applying the process and often lead them into unnecessary work.

2nd Activity: The students were given the following augmented matrix of a system of linear equations (Table 1) and they were asked: "what can you deduce about the original system? What do you assume about its solutions?"

Table 1: Augmented matrix of a system of linear equations

3 5 1	1
0 0 0	0

At first, the students answered that the system was indeterminate, an answer which in the general context of our question is not wrong. However, it was wanted from them to specify with respect to the notional part, i.e., what kind of indeterminacy we have here. The students appeared to be hesitant and they were prompted to write down the equivalent system equations and phrase better their conclusions. After the appropriate process, they found the general parametric solution of the system, yet some of them concluded: "so I do not have indeterminacy", "I have a line". Other students, continuing the examination of the systems that they extracted the formula that gives its infinite number of solutions, also arrive at the conclusion that the system degenerates to the equation of a straight line and the solutions have the parametric form of the points of that line.

What puzzled us is that certain students did not connect the indeterminacy, as a characteristic of the system, with the existence of the line as its solution. The spontaneous responses indicate the type of confusion they have at the level of characterization of a system's solutions, involving the everyday meaning of the Greek word for *indeterminacy* with the mathematical term. We conclude that they appear to lack the conceptual understanding of something essential, i.e., that with the phrase "unique solution", irrespectively of the space in which we search for solutions of a system (R^2 or R^3), we mean one point and not one unique line or one unique plane. They regard the solution of the system as a process, so initially they call it "indeterminate". The algebraic result for the solution puzzles them, because they have an incomplete perception or understanding of the graphical representation. The difference between the system of equations and the system of lines that has been used to interpret the former allows them, through the counterexample offered in the form of the augmented matrix, to express their confusion, a fact that more or less demonstrates their independent approach of the equivalent representations.

***3rd* Activity:** To the question: "When two systems of linear equations are called equivalent? What about the equations of the equivalent systems: should be identical or not?", most students answered correctly that they are equivalent when they have the same solutions, and that "it is not necessary for the equations of the two systems to be identical." To the next question, "is there any restriction to the number of systems that are equivalent to a given system?", a need arose for a kind of indirect clarification, since that question was not clear. Therefore, the case of the existence of a unique solution of linear systems was offered for investigation, with a specific reference to its graphical representation; in addition, the students were asked to draw the respective graphs, when this was possible. With respect to the geometrical representation of the unique solution for a 2×2 system, they correctly answered that it is "a point, the intersection of two lines". In their answers to the question "how could we represent an equivalent system in the same graph", they were also correct:

- "with another intersection of two lines that will pass from the same point of intersection",
- "another pair of lines" and
- "is it not possible with an additional line?".

Thus, a variety of representational constructions is recorded for the systems in R^2 that can be characterized as equivalent with respect to a given system.

However, an essential difference was noticed. Certain students apparently assumed that, since the original system consists of two lines, its equivalent should also have two intersecting lines at the common point of the first two lines. Other students showed with their answer that they understand much better the concept of equivalence, since, as they correctly pointed out, with the addition of only one line that passes from the common point of the other two, a new system of three lines is formed, as well as sub-systems that have as a solution the unique solution of the initial system. This idea indicates a more abstract way of thinking and verifies for these students a higher quality of visualizing the object dimension of the "embodied world of mathematics", while at the same time indicates versatility in their part, in the formal (axiomatic) world. Eventually, a key aspect of the notion of equivalence became clear to all students: that it does not depend on the equality of the number of lines, describing the original and the new system, but solely on the identical solutions these must have.

An essential problem arose when the corresponding process was examined for a 3×3 system of linear equations with a unique solution. At first, it was asked what should be added in order for the system to keep its single-solution status, to which the students correctly answered that they should add planes. However, our next question revealed another misunderstanding: when asked "since it should be unique, what should be the geometrical representation of the solution [to the 3×3 system]?", they responded "a line!" Thus, they essentially ruled out the case that the unique solution to a 3×3 system is a point, as the intersection of three planes. This strengthens our initial conclusion that there is confusion with respect to the uniqueness of the solution of a linear system in the space R^3. The essential problem, faced here, has to do with the conceptual understanding of the uniqueness of the solution. It appears that there is a gap between the algebraic expression of this uniqueness and its graphical representation, with confusion being observed at three levels: 1) the solution level, 2) the nature of the solution, and 3) the characterization of the system.

As far as the comprehension level of the students is concerned, it is concluded that presumably they had not "embodied" in the object dimension what means infinitely many solutions, no solution or a unique solution in the case of the 3×3 system. The students appeared to confuse the notion of equivalence with the notion of dimensions, i.e., they supposed that by adding a dimension they would have the solution following this expansion, thus converting the point-solution that corresponds to a 2×2 system to a line in the three-dimensional space, as the unique solution to a 3×3 system. In this way, it is revealed that they lack the ability to make the correct reduction in the "process". The students have not generalized in an abstract way, matching examples from their physical environment (such as the unique point as the intersection of two walls with the roof of a room); thus, they attempt to make representations using the wrong generalization. According to the "three worlds" approach, the issue of the existence or not of experience is not exclusively or primarily intrinsic to the subject, but a real issue is posed with respect to the external requirements for representation. The internalization of the structural elements and terms, which according to Tall allows the transition from one "world of mathematics" to the other two, is a prerequisite for the ability to develop and use relative representations. In our case, there are mathematical and physical representations, which have not been embodied into a single object, a fact resulting in difficulties in shifting from one representation to the other.

***4th* Activity:** *The following equivalent systems of linear equations* (Table 2) *are given:*

Table 2: Equivalent systems of linear equations

$x_1+3x_2+5x_3=7$	$x_1+3x_2+5x_3=7$	$x_1-7x_3=-14$
$3x_1+7x_2+7x_3=7$	$-2x_2-8x_3= -14$	$x_2+4x_3=7$
$x_2+4x_3=7$	$x_2+4x_3=7$	

A solution to the first system is (x_1, x_2, x_3) = (0,-1,2). Is this also a solution to the other two? Why?

In this case, we are interested in examining the ability of the students to interpret and apply the properties of the concept of equivalent systems in the specific context. The action seeks new fields in order to extract conclusions, such as the embodiment of systems (with respect to their geometrical representation). One group of students made use of the information that the systems are equivalent and answered that, since this holds true, the solution to the first will also be the solution to the other two. Another students' group, although also resorted to the notion of equivalence as a sufficient element of reasoning, wanted to verify their rationale by substituting the ordered triad in the equations of the three linear systems.

During the investigation, interesting views were expressed by various students, such as that the line operations are the ones that can transform one system to another, which will be equivalent to the original one. This fact indicates a certain degree of encapsulation of the process of operations with matrix elements. In the next step, attracting the attention of the students to the solution set, the following dialogue took place (between Instructors "I" and Students "S"):

- I: "By examining the third system, knowing that all three are equivalent, could you extract a conclusion about the solutions to the systems?"
- S: "We will have a certain indeterminacy for the solutions to the first and second system"
- I: "And what will we have for the third one?"
- S: "I will also have indeterminacy for this, because they are equivalent".

We would like to point out that the indeterminacy of the third system was produced as a result of the equivalence relation among all three systems and not from its separate investigation. Probably the view of a 2×3 system (that differs from the typical algebraic form of a 3×3) puzzled the students in their perception of the relative positions of the unknown quantities. Of course, if they were more familiarized with the algebraic representation, the form of the third system should be the best evidence of the indeterminacy of the three equivalent systems. When we insisted in the first question: "If we examined the third system separately, from where do we understand that we have indeterminacy?", the students answered:

- "I have three variables and two equations".

"Yes, but when I have more unknown variables than equations, I do not know whether it is indeterminate or inconsistent".

- "Indeed, there are two possibilities; however, since we are given one solution, it is indeterminate."

The answers appear to be intuitional and they are fully formulated only with their collective interaction, which resulted in a more complete and clear reasoning.

The rationale is reasonable and appears to have been present from the start in their complementary answers, but without being fully shaped up to the point of this small conversation. Thus, to the question: "This point that is given to us as a solution, where should it belong to?", they answer "to the intersection of the planes", "that is, to one line". It is interesting here that the line, as the set of solutions to the third system, results from abstract thought, which again is collective. The students realize the multiple natures of the data and through their mutual feedback they also realize the existence of indeterminacy, thus characterizing the third linear system. Search for further information is done in the three systems of linear equations and the relation among them, by centering the attention of the students to the representations, both to the geometrical and the form of the augmented matrix for each system. By writing down the matrix form of the third system of equations, the students conclude that the matrix is the outcome of a diagonalization and, more specifically, that it is an upper triangular matrix. However, the students wanted to test these conclusions in the multi-representational computational environment Mathematica (Fig. 2), in order to "see the indeterminacy", as they said characteristically.

Conclusively, the students stress the indeterminate character of the third system and they wonder, given the equivalence of the systems, about the possible form that the diagonal matrices of the first two systems should have; then they pursue their calculation. To this end, they continued to use *Mathematica*, making use of the automations that provides.

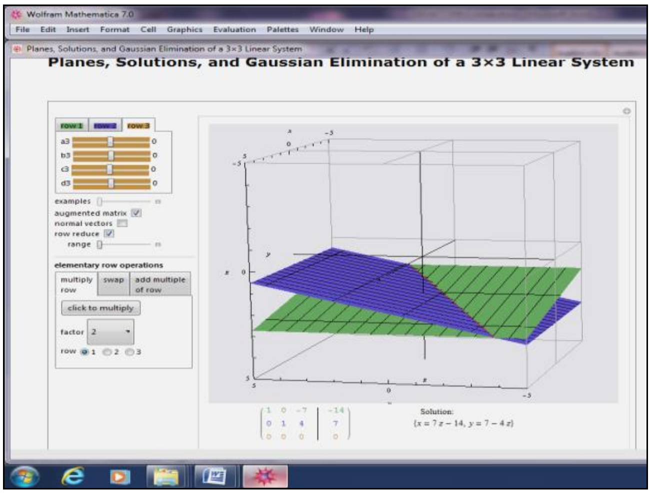

Fig 2: The multi-representational computational environment Mathematica.

However, before doing operations in the context of the Gaussian elimination, in an intermediate question on the geometrical representation of each system (save from the set of solutions), the students make a mistake. Apparently, due to either lack of observing ability or lack of focus, they do not notice that in the second system the second and the third equation are multiples of each other; therefore, they express the opinion that each of the first two systems corresponds to three planes that intersect each other, while the third system is represented by two intersecting planes. After they input the data of the systems into *Mathematica*, they realize that the second system corresponds essentially to two and not three intersecting planes. The resulting cognitive conflict led them to seek the reason for the contradiction, and some of the comments heard were:

- "here I lack a plane!",
- "it seems that we wrote something wrong",
- "three equations…two planes?!",
- "let's check again the equations",
- "some of them would rather be identical".

Here one can witness the contribution of the dynamic environment, both in the "decoding" of the algebraic form in geometrical terms and in pinpointing the "hidden"

properties of the systems, put there by design. This "revelation" tested the observing ability of the students by producing a cognitive conflict with the notion of equating the number of equations with the number of planes in space, and emphasizing the special case of two coinciding planes.
From the brief dialogues quoted above, it appears that the intervention of the technological environment facilitates learning, through visual representations of concepts and because it allows the students to observe them and think about them in a more active way. Moreover, by automating the computation of the Gaussian elimination, the students were able to employ the software environment, in order to realize quickly and easily that the third system is essentially the triangular form, assumed by the first two systems after their diagonalization. In this way, they have more time available to reflect on the nature and the meaning of the process, instead of simply involving themselves with the relative calculations. Thus, the documented transcendence of the procedural knowledge level is to the benefit of a deeper understanding of the concepts and the relations between them.

6. Discussion and Conclusions

The findings of this study are analyzed with a systematic description of the learning difficulties that were detected, with an interpretation and classification of the mathematical understanding by the students, in the categories of the analytical tools that were mentioned and with a following of the evolution of this understanding. Answers are given to the research questions and finally an interpretation is offered, along with a discussion on the findings of this study.

6.1 Research question I

In the context of the instructional intervention, a series of specific difficulties were detected, among which the most important ones were the following:

- A relative difficulty in transforming the equation of a line into a parametric form was recorded. This difficulty has to do with the limited familiarity of the students with the three-dimensional mathematical space, for which some students could not render the line equation.
- A difficulty to disconnect the dimension of a system from the dimension of space. This obstacle can be traced to an unsuccessful transfer of cognitive schemes from the framework of some previous knowledge.
- A difficulty of writing down the matrix equation form of a system, starting from its typical form. A wrong transfer was made from the form of the augmented matrix to the form AX = B. It appears that there was incomplete connection with the rules of matrix multiplication.
- A difficulty in transition between equivalent symbolic expressions as interrelated ones. The correspondence between forms presented difficulty, even between forms with which the students were familiar with.
- Treating processes as black boxes. For example, in the students' perception of the method of Gaussian elimination, the drawing of conclusions (e.g., for the existence of a unique solution) would follow the completion of the transformation to the reduced form. A result of this was the useless application of the whole algorithm, in cases of obvious inexistence of solution or of indeterminacy.
- A difficulty to discriminate between a system of equations and a system of lines (and even more a system of planes). The particular difficulty seriously hindered the ease with which the students represented the system in geometrical terms. Here, there is another indication of non-embodied visual representation.
- Incapability, in general, to connect representations: they were not perceived as equivalent different expressions at all.

Limited ability of representation in the three-dimensional space. Confusion was created by the idea that the unique solution to a 3×3 system could be a point of intersection of three planes. Although the students could understand the solutions as points algebraically, they could not do so geometrically. It appears that there is a gap between the algebraic expression of the uniqueness of solution and its graphical representation. Thus, it was revealed that the conceptual understanding of the solution's uniqueness in the three-dimensional space was incomplete; this is related to the aforementioned difficulty to embody a geometrical representation in that space.

6.2 Research question II

At a more theoretical level, we can see here the development of visual-spatial skills of the students in combination with the symbolic, algebraic and matrix approach of the problems or with the additional connection of embodied with symbolic knowledge [20, 21]. In the terms of Tall's theory, starting from the formal world and having only a typical knowledge of the definitions, the students, through these activities, move and process the concepts in the embodied world, creating knowledge that allows for their symbolic handling and eventually the return at the level of the typical language of the axiomatic world with a new depth in knowledge. Therefore, what it is observed is the development of a more rich and coherent mental image for the concept of equivalence in linear systems, based on the manner these systems are examined with, and at the same time the development of the students' ability to transfer, through investigative activities, the new knowledge to different frames. These processes appeared to lead to the encapsulation (if not to the embodiment) of the symbolic language, as predicted by Tall [10].
This scaffolding set of activities has eventually a significant meta-cognitive value. Through the cognitive conflict caused, the concept of linear system assumes a novel meaning for the students, while around it, new conceptual interconnections and new knowledge schemes are developed [10]. The students appeared to fully comprehend (albeit *a posteriori*) the manner in which the actions operated for them.
Also, the power of the collective process was brought out, when such a process takes place within a certain frame suitably organized. The work done by some students probably would not have emerged without the observations of other students. Given the nature of the two contributions, one could speak about *collaborative instrumental genesis* and more specifically in the highest-level, that of *collaborative instrumentation*.
Conclusively, a crucial factor for the acquisition of learning benefits from the multi-representational environment is its pedagogical exploitation and not its mere presence in the

learning process. Consequently, the crucial nature of the approach and methodology of instruction is verified. The theoretical framework, developed combining Tall and Kolb's approaches, allows for the maximization of benefits and renders the use of the environment constructive. Furthermore, multiple applications are needed – and thus crossings of both the instructional cycle and the "three worlds" – in order to achieve a satisfactory level of experience and ability to transfer from one representation to the other; something that was a basic characteristic of this instructional intervention.

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